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# OPEN PROBLEMS IN PRIME NUMBER THEORY

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*ABSTRACT This paper investigates the theoretical architecture of the main unsolved puzzles in prime number theory. It analyzes fundamental mathematical barriers, such as the parity problem for twin numbers, the connections between complex analysis and quantum mechanics, and the impact of computational algorithms.*

**Keywords**: prime number, Riemann

## INTRODUCTION

Prime numbers are the foundation of the structure of natural numbers. It is observed that, although their definition is elementary, their distribution exhibits a fascinating duality: an appearance of a stochastic process on a small scale, doubled by a rigorous statistical regularity on a large scale, governed by logarithmic functions. Historically, their study forced the transition from Euclid's elementary arithmetic to analytical methods of extreme complexity. Today, the theory of prime numbers is no longer just a mathematical curiosity, but the backbone of global information security.

## I. The Great Conjectures: Foundations and Recent Advances

### 1. The Riemann Hypothesis: The Horizon of Complex Analysis and the Quantum Connection

The Riemann Hypothesis (1859) is considered to be the deepest open problem in mathematics. The hypothesis states that all the "nontrivial" zeros of the Riemann zeta function ( $\zeta(s)$) lie on a single vertical line in the complex plane, called the critical line, where the real part of the complex number $s$ is exactly $1/2$

Bernhard Riemann's Biographical Background

Riemann, a student of Gauss at Göttingen, worked at a time when mathematics was moving from geometric intuition to the rigor of analysis. His premature death left unfinished the proof

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that would have provided absolute control over the error term in the Prime Number

**Montgomery-Odlyzko Connection**

A fundamental discovery of the 20th century is the connection between Riemann zeros and quantum mechanics. It has been shown that the spacing of these zeros follows a distribution identical to the eigenvalues of random Hermitian matrices (GUE). This analogy, known as the Montgomery-Odlyzko Distribution, suggests that the zeros could represent the energy levels of an as-yet-unknown chaotic quantum system[1].

## 2. Goldbach's Conjecture: From the Euler to Helfgott Correspondence

Proposed in 1742 through correspondence between Christian Goldbach and Leonhard Euler, the conjecture states that any even number n>2 is the sum of two prime numbers.

1. Computational verification algorithms

The verification effort used algorithms based on the optimized Sieve of Eratosthenes and Fast Fourier Transforms (FFT) to speed up the calculation of sums. Currently, the verification limit has reached the threshold of $4 * 10^{18}$, without any counterexamples being found.

2. Recent Results

A historic breakthrough was made in 2013, when Harald Helfgott proved the "weak" (ternary) version of the conjecture. This analytically confirms that any odd number n>2 is the sum of three prime numbers, eliminating the need for computational verifications for this class of numbers[2].

## 3. The Twin Prime Conjecture and the Parity Barrier

The problem of the existence of an infinite number of pairs (p,p+2) is one of the most persistent challenges in sieve theory.

The mathematical foundation of the parity barrier

It is noted that traditional sieve methods (Selberg, Brun) suffer from an intrinsic limitation: they cannot distinguish between numbers that have an even number of prime factors and those that have an odd number. This "parity problem" has blocked progress for a

[1] Sarnak, P. (2011). "Statistical Properties of Zeta Zeros". Clay Mathematics Institute.
[2] Helfgott, H. A. (2013). "Major arcs for Goldbach's theorem". arXiv:1305.2897.

century, because the sieve produces "near-primes" instead of pure primes.

The Maynard-Zhang leap

A revolution occurred in 2013, when Yitang Zhang circumvented the parity barrier by demonstrating a finite distance (d<70 million) between an infinity of primes. Later, James Maynard refined weighted sieve techniques, reducing this limit to 246[3].

## II. Niche issues and less explored frontiers

In addition to the great conjectures, prime number theory includes hypotheses of remarkable structural finesse:

**Gilbreath's conjecture (1958)**: Postulates an absolute regularity in the successive differences of primes, where the first term of each series of differences remains constant 1.

**Firoozbakht's conjecture (1982)**: Suggests that the distance between primes grows much more slowly than classical models indicate, proposing a strict decrease in the ratio $\sqrt[n]{p_n}$ [4].

**Oppermann's conjecture**: Affirms the certain presence of primes in short intervals of the type $[n^2 - n, n^2]$, providing a local density much higher than that predicted by probabilistic models.

## III. The contribution of Romanian mathematicians

The Romanian school of mathematics has historically shown a deep interest in number theory, managing to create bridges between classical analysis and modern algebraic structures. The contribution of Romanian researchers is not just a niche one, but directly targets the asymptotic behavior and density of prime numbers.

1. **Traian Lalescu and the foundation of asymptotic sequences**

It is noted that one of the most famous early contributions belongs to Traian Lalescu (1882–1929). He proposed the study of the limit of the sequence $L_n = \sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n}$ , where $p_n$ represents the n-th prime number. Although it seems like an elementary analysis problem, the “Lalescu sequence” is intrinsically linked to the distribution of primes and to the error in the Prime Number Theorem. His research demonstrated that this difference tends to zero, providing an early insight into the

[3] Maynard, J. (2014). "Small gaps between primes". Annals of Mathematics, Vol. 181, Issue 1.
[4] Ribenboim, P. (2004). The Little Book of Bigger Primes. Springer-Verlag.

regularity of the increase of primes at infinity[5].

### 2. Dan Barbilian and the axiomatization of numerical structures

Known in the literature under the pseudonym Ion Barbu, Dan Barbilian brought an algebraic perspective to number theory. He explored number rings and their arithmetic, laying the foundations for geometric structures that reflect fundamental numerical properties. His works on the metrization of spaces and the theory of algebras have influenced the way in which Romanian researchers today approach problems of the distribution of primes through the prism of number geometry.

### 3. Nicolae Popescu and the theory of fields

An important contribution belongs to Nicolae Popescu, who, through his treatise "Theory of Numbers", offered a modern synthesis of analytical and algebraic methods. His work facilitated the formation of a generation of mathematicians who today use the theory of number fields to investigate conjectures such as Artin's or the distribution of primes in arithmetic progressions.

## IV. Computational Impact and the Post-Quantum Future

Technology has transformed number theory from a purely abstract discipline into an applied one. Distributed computing projects (such as GIMPS) are testing the limits of primality, but the future of security is threatened by quantum computers.

Shor's algorithm demonstrates that factoring large numbers (the basis of RSA) can be done in polynomial time. There is an accelerated migration towards Post-Quantum Cryptography (PQC), based on mathematical lattices, where prime numbers are no longer the only pillar of defense, but become component elements in much more complex multidimensional geometric structures[6].

## Conclusions

The open problems in prime number theory represent the current limit of human knowledge. The persistence of these enigmas, despite massive technological advances, underscores the unfathomable depth of the arithmetic universe. The search for order in the sequence of prime numbers remains, without a doubt, the purest form of intellectual exploration of humanity.

[5] Lalescu, T. (1900). "Sur une suite de nombres". Gazeta Matematică.

[6] NIST (2024). "Post-Quantum Cryptography Standardization". National Institute of Standards and Technology.

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