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TENSORIAL FORM OF THE EQUATIONS OF MOTION

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Abstract. In this paper we will establish the evolution equations of closed relativistic systems which represent a generalization of the Euler-Lagrange equations and we will modify the form of the equations of motion so that they appear as conservation laws, finally obtaining the tensor form of the equations of motion.

Keywords: tensorial forms, equations of motion, Minkowski space

1. INTRODUCTION

To begin with, we will write the equations of motion of a relativistic system with the help of a relativistic Lagrangian, in which instead of time all the coordinates intervene x^0, x^1, x^2, x^3.

The variational method of studying a physical system consists in finding a function $L(q_k, \dot{q}_k. t)$, called the Lagrangian of the system, so that the real evolution of the system is exactly the minimum of the action

$$S = \int_1^2 L(q_k, \dot{q}_k. t)dt \ . \qquad (1)$$

This minimum is described by the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \ . \qquad (2)$$

It is worth noting that closed systems are characterized by the fact that L does not depend explicitly on time.

Next, we will extend the notion of Lagrangian to the case when the generalized coordinates do not depend only on t, but on the four relativistic variables $x^0 = ct, x^1 = x, x^2 = y$ and $x^3 = z$, and the action is obtained as an integral over a 4-dimensional domain in Minkowski space.

The relativistic, or 4-dimensional, Lagrangian will be a function of the form $L^*(q_k, \dot{q}_{kv}, x^0, x^1, x^2, x^3)$ where q_k are the generalized coordinates of the considered system, depending on x^0, x^1, x^2, x^3, and by notation,

$$\dot{q}_{kv} = \frac{\partial q_k}{\partial x^v}\ , v = 0,1,2,3.$$

The dependence of the quantities q_k and $\dot{q}_{kv}$ on spatio-temporal coordinates x^μ is assumed to be smooth, so we can consider the relations fulfilled

$$\frac{\partial \dot{q}_{k\mu}}{\partial x^v} = \frac{\partial \dot{q}_{kv}}{\partial x^\mu}\ ,\quad \mu, v = 0,1,2,3. \qquad (3)$$

The number of generalized coordinates q_k may or may not be finite. By analogy with closed mechanical systems, we will consider a relativistic physical system (described in Minkowski space) to be closed if its 4-dimensional Lagrangian does not explicitly depend on x^0, x^1, x^2 and x^3 .

The 4-dimensional relativistic action of a relativistic system is defined by the formula

$$S = \frac{1}{c}\int_\Delta L\left(q_k, \dot{q}_{k\mu}, x^0, x^1, x^2, x^3\right) dx^0 dx^1 dx^2 dx^3\ , \qquad (4)$$

where the integral extends over a domain Δ in the Minkowski space $\mathbb{R}^4$ and usually has the form $\Delta = [t_1, t_2] \times \mathbb{R}^3$. For brevity we will denote the 4-dimensional volume element by $dV^* = dx^0 dx^1 dx^2 dx^3$, so if dV is the 3-dimensional volume element, we have $dV^* = c\ dt\ dV$.

Formula (4) will be written in the form

$$S = \frac{1}{c}\int_\Delta L^* c\ dt\ dV = \int_{t_1}^{t_2} dt \int_{\mathbb{R}^3} L^* dV\ , \qquad (4')$$

where it is seen that:

$$L = \int_{\mathbb{R}^3} L^* dV$$

plays the role of the Lagrangian in the formula (1). This observation justifies the name Lagrangian density given to the relativistic Lagrangian L^*. It can also be seen that the factor $\frac{1}{c}$ was introduced in (4) precisely so that $(4')$ would reduce to (1).

Next we will establish the evolution equations of closed relativistic systems, which represent a generalization of equations (2) for action (1). For this we will take into account that in the case of closed relativistic systems, the expression (4) of the 4-dimensional action becomes

$$S = \frac{1}{c}\int_{\Delta} L^*\left(q_k, \dot{q}_{k\mu}\right) dV^* \ . \qquad (5)$$

Theorem.(Euler-Lagrange equations for relativistic systems). The evolution equations of any closed relativistic system, with the Lagrangian L^*, are

$$\frac{\partial L^*}{\partial q_k} - \sum_{v=0}^{3} \frac{\partial}{\partial x^v}\frac{\partial L^*}{\partial \dot{q}_{kv}} = 0 \ . \qquad (6)$$

as many in number as there are generalized q_k coordinates.

2. TENSORIAL FORM OF THE EQUATIONS OF MOTION

Next we will modify the form of the equations of motion so that they appear as conservation laws.

We will amplify each of the equations (6) by $\dot{q}_{k\mu}$ and sum after k (assuming that the respective series are convergent, this in case there is an infinity of quantities q_k), obtaining:

$$\sum_k \frac{\partial L^*}{\partial q_k}\dot{q}_{k\mu} - \sum_{k,v} \dot{q}_{k\mu}\frac{\partial}{\partial x^v}\frac{\partial L^*}{\partial \dot{q}_{kv}} = 0 \ .$$

If we take into account the product derivation rule, the last sum above is written

$$\sum_{k,v} \dot{q}_{k\mu}\frac{\partial}{\partial x^v}\frac{\partial L^*}{\partial \dot{q}_{kv}} = \sum_{k,v}\frac{\partial}{\partial x^v}\left(\dot{q}_{k\mu}\frac{\partial L^*}{\partial \dot{q}_{kv}}\right) - \sum_{k,v}\frac{\partial L^*}{\partial \dot{q}_{kv}}\frac{\partial \dot{q}_{k\mu}}{\partial x^v} \ ,$$

so the previous equation becomes:

$$\sum_k \frac{\partial L^*}{\partial q_k}\dot{q}_{k\mu} - \sum_{k,v}\frac{\partial}{\partial x^v}\left(\dot{q}_{k\mu}\frac{\partial L^*}{\partial \dot{q}_{kv}}\right) - \sum_{k,v}\frac{\partial L^*}{\partial \dot{q}_{kv}}\frac{\partial \dot{q}_{k\mu}}{\partial x^v} = 0 \ .$$

If in the last sum we replace (according to (3)):

$$\frac{\partial \dot{q}_{k\mu}}{\partial x^v} = \frac{\partial \dot{q}_{kv}}{\partial x^\mu}$$

and we keep in mind that:

$$\frac{\partial L^*}{\partial x^\mu} = \sum_k \frac{\partial L^*}{\partial q_k}\frac{\partial q_k}{\partial x^\mu} + \sum_{k,v}\frac{\partial L^*}{\partial \dot{q}_{kv}}\frac{\partial q_{kv}}{\partial x^\mu}$$

we will be able to write the equation deduced from the equations of motion in a simpler form

$$\frac{\partial L^*}{\partial x^\mu} - \sum_{k,v} \frac{\partial}{\partial x^v}\left(\dot{q}_{k\mu}\frac{\partial L^*}{\partial \dot{q}_{kv}}\right) = 0 .$$

These four equations (corresponding to the values $\mu = 0,1,2,3$) are nothing but another form of the motion equations (6). So that the derivatives with respect to x^μ and those with respect to x^v, no longer appear in these equations, we will write

$$\frac{\partial L^*}{\partial x^\mu} = \sum_{v} \frac{\partial}{\partial x^v}\left(\delta_\mu^v L^*\right) .$$

so that the last form of the equations of motion becomes

$$\sum_{v} \frac{\partial}{\partial x^v}\left(\sum_{k} \dot{q}_{k\mu}\frac{\partial L^*}{\partial \dot{q}_{kv}} - \delta_\mu^v L^*\right) = 0 . \qquad (7)$$

If we take into account the tensorial character of the quantities whose components are partial derivatives, it follows that the expressions written in the brackets in formula (7) are the components of a mixed tensor, which we will write

$$\tilde{T}_\mu^v = \sum_{k} \dot{q}_{k\mu}\frac{\partial L^*}{\partial \dot{q}_{kv}} - \delta_\mu^v L^* . \qquad (8)$$

Writing equations (7) using notation (8), we obtain **the tensor form of the equations of motion**:

$$\sum_{v=0}^{3} \frac{\partial \tilde{T}_\mu^v}{\partial x^v} = 0 , \quad \mu = 0,1,2,3. \qquad (9)$$

Since the left-hand member of these equations is a 4-divergence, we can consider these equations as conservation laws for the quantities

$$T_\mu = \left(T_\mu^0, T_\mu^1, T_\mu^2, T_\mu^3\right), \mu = 0,1,2,3 .$$

BIBLIOGRAPHY

1. **Anderson J**., *Principes of relativity Physics*, Academic Press,1967

2. **Barbu V.,** *Metode matematice în optimizarea sistemelor diferențiale*. Editura Academiei, București, 1989.
3. **Callahan J.**, *The Geometry of Spacetime*, Springer-Verlag, 2000.
4. **Courant R., Hilbert D.**, *Methods of mathematical physics. Vol. II: Partial differential equations*. (Vol. II by R. Courant.). Interscience Publishers, New York-London, 1962
5. **Einstein A**., *Teoria Relativității*, Ed. Tehnică, București, 1957.
6. **Naber G**., *The Geometry of Minkowski Spacetime*, Springer-Verlag, 1992.
7. **Zeeman E.C.,** *The topology of Minkowski space, Topology*, vol. 6 (1967).
8. **Taylor E., Wheeler A.** , *Spacetime Physics: Introduction to Special Relativity*. W. H. Freeman. 1992.