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INVARIANCE OF MAXWELL'S EQUATIONS TO LORENTZ TRANSFORMATIONS IN MINKOWSKI SPACES

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Abstract. *In this paper we consider known that an electromagnetic field is described by two fields: an electric field and a magnetic field. Between these fields there are certain relationships, known as Maxwell's equations. One problem that can be asked is how to change these fields by switching from one inertial observer to another. We will try to formulate an answer based on the invariance of Maxwell's equations with respect to Lorentz transformations. In this sense we will state a theorem that shows how these fields change. Finally, we present some consequences of this theorem.*

Keywords: Minkowski spaces, Maxwell's equations, electromagnetic field

1. INTRODUCTION

We consider it known that an electromagnetic field is described by two fields, one denoted $\overline{E}:D\to R^3$, called electric field and another $\overline{H}:D\to R^3$ called a magnetic field. Here D is a domain in the R^4 event space, namely $\overline{E}$ and $\overline{H}$ they have four variables each, certain $t=¿$ temporal variable and $x,y,z=¿$ spatial variables. Having values in R^3, these fields have three components each, noted E_x,E_y,E_z and respectively H_x,H_y,H_z. The following relations, known as Maxwell's equations, take place between these fields.

$$\frac{\partial\overline{D}}{\partial t}=rot\,\overline{H}\,(1)$$

$$\frac{\partial\overline{B}}{\partial t}=-rot\,\overline{E}\,(2)$$

$$¿\,\overline{D}=0\,(3)$$

$$¿\,\overline{B}=0\,(4)$$

$$\overline{D}=\varepsilon_0\overline{E}\,(5)$$

$$\overline{B}=\mu_0\overline{H}\,(6)$$

From the form in which these equations were written here it is recognized that the environment considered is vacuum, without electrical charges and without currents. These equations are written for a fixed Ω observer, the fields $\overline{E}$ and $\overline{H}$ having other values for another inertial observer ω moving with respect to Ω. One problem that can be asked is how to change these fields by switching from one inertial observer to another. We will try to formulate an answer based on the invariance of Maxwell's equations with respect to Lorentz transformations and for the concrete solution of the problem we will limit ourselves to the special Lorentz transformations

$$\begin{cases} x' = \dfrac{x - vt}{\sqrt{1 - \dfrac{v^2}{c^2}}} \\ y' = y \\ z' = z \quad (7) \\ t' = \dfrac{t - \dfrac{v}{c^2}x}{\sqrt{1 - \dfrac{v^2}{c^2}}} \end{cases}$$

corresponding to a particular choice of coordinate axes in relation to the direction of relative movement of observers. In addition to the invariance of Maxwell's equations with respect to Lorentz transformations, the principle of relativity will also be used through its consequences on the constancy of the parameters ε_0 and μ_0 as well as the invariance of the electric charge following the transition from one inertial observer to another.

2. INVARIANCE OF MAXWELL'S EQUATIONS TO LORENTZ TRANSFORMATIONS

We will further state a theorem in which we will express the invariance of Maxwell's equations.

Theorem. If an electromagnetic field with respect to the inertial observer Ω has the components E_x, E_y, E_z, respectively H_x, H_y, H_z and ω is another inertial observer that moves with respect to Ω with velocity v along the axis $Ox = Ox'$, then in the system ω he components of this field are

$$E'_x = E_x$$

$$E'_y = \frac{E_y - v\mu_0 H_z}{\sqrt{1-\frac{v^2}{c^2}}} \quad (8)$$

$$E'_z = \frac{E_z - v\mu_0 H_y}{\sqrt{1-\frac{v^2}{c^2}}}$$

respectively

$$H'_x = H_x$$

$$H'_y = \frac{H_y - v\varepsilon_0 E_z}{\sqrt{1-\frac{v^2}{c^2}}} \quad (9)$$

$$H'_z = \frac{H_z - v\varepsilon_0 E_y}{\sqrt{1-\frac{v^2}{c^2}}}$$

Demonstration. In the landmark of the observer Ω the equations (1), (2), (3) and (4) in which we take into account and (5) and (6) are written in the form

$$\begin{cases} \varepsilon_0 \frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \varepsilon_0 \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \varepsilon_0 \frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{cases} \quad (1')$$

$$\begin{cases} -\mu_0 \dfrac{\partial H_x}{\partial t} = \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} \\ -\mu_0 \dfrac{\partial H_y}{\partial t} = \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} \\ -\mu_0 \dfrac{\partial H_z}{\partial t} = \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} \end{cases} (2')$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 (3')$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0. (4')$$

To simplify writing we will note the Lorentz-Fitzgerald factor with

$$\alpha = \sqrt{1 - \frac{v^2}{c^2}} \wedge \gamma = \alpha^{-1}.$$

We perform the demonstration in two stages

<u>Stage I.</u> We prove the first formula in (8) and the last two formulas in (9). If in the expression:

$$\frac{\partial E_x}{\partial t} = \frac{\partial E_x}{\partial t'} \frac{\partial t'}{\partial t} + \frac{\partial E_x}{\partial x'} \frac{\partial x'}{\partial t} + \frac{\partial E_x}{\partial y'} \frac{\partial y'}{\partial t} + \frac{\partial E_x}{\partial z'} \frac{\partial z'}{\partial t}$$

we replace, according to (7),

$$\frac{\partial t'}{\partial t} = \gamma, \frac{\partial x'}{\partial t} = -v\gamma, \frac{\partial y'}{\partial t} = \frac{\partial z'}{\partial t} = 0$$

result

$$\frac{\partial E_x}{\partial t} = \gamma \left(\frac{\partial E_x}{\partial t'} - v \frac{\partial E_x}{\partial x'} \right).$$

Proceeding by analogy, we can express the derivative in relation to x',

$$\frac{\partial E_x}{\partial x'} = \frac{\partial E_x}{\partial t} \frac{\partial t}{\partial x'} + \frac{\partial E_x}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial E_x}{\partial y} \frac{\partial y}{\partial x'} + \frac{\partial E_x}{\partial z} \frac{\partial z}{\partial x'}$$

where, according to the inverses of the formulas (7) we have

$$\frac{\partial t}{\partial x'} = \gamma \frac{v}{c^2}, \frac{\partial x}{\partial x'} = \gamma, \frac{\partial y}{\partial x'} = \frac{\partial z}{\partial x'} = 0.$$

We get it that way

$$\frac{\partial E_x}{\partial x'} = \gamma \left(\frac{\partial E_x}{\partial x} + \frac{v}{c^2} \frac{\partial E_x}{\partial t} \right)$$

which replaced above leads to the equation

$$\frac{\partial E_x}{\partial t}=\gamma\frac{\partial E_x}{\partial t'}-\gamma^2 v\left(\frac{\partial E_x}{\partial x}+\frac{v}{c^2}\frac{\partial E_x}{\partial t}\right).$$

After a simple calculation from here we can determine

$$\frac{\partial E_x}{\partial t}=\alpha\frac{\partial E_x}{\partial t'}-v\frac{\partial E_x}{\partial x},(¿)$$

which, if we take into account $(3')$, becomes

$$\frac{\partial E_x}{\partial t}=\alpha\frac{\partial E_x}{\partial t'}+v\left(\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)$$

Replacing this expression in the first formula of $(1')$ and taking into account that $y=y'$ and $z=z'$, is obtained:

$$\varepsilon_0\frac{\partial E_x}{\partial t'}=\frac{\partial}{\partial y'}\left[\gamma\left(H_z-v\varepsilon_0 E_y\right)\right]-\frac{\partial}{\partial z'}\left[\gamma\left(H_y+v\varepsilon_0 E_z\right)\right].(10)$$

On the other hand, according to the principle of relativity, if Maxwell's equations have the same form for the observer ω, the first formula in $(1')$ will be written:

$$\varepsilon_0\frac{\partial E_x'}{\partial t'}=\frac{\partial H_z'}{\partial y'}-\frac{\partial H_y'}{\partial z'}$$

Comparing this expression with (10), the searched formulas are obtained.

<u>Stage II.</u> We prove the first formula in (9) and the last two formulas in (8). To this purpose note that the formula (¿) was obtained only on the basis of the calculation of some derivatives of compound functions, using formulas (7), so it can also be written for H_x, e:

$$\frac{\partial H_x}{\partial t}=\alpha\frac{\partial H_x}{\partial t'}-v\frac{\partial H_x}{\partial x}$$

Taking into account $(4')$, it results:

$$\frac{\partial H_x}{\partial t}=\alpha\frac{\partial H_x}{\partial t'}+v\left(\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right).$$

Which replaced in the first formula $(2')$, leads to the next relation

$$-\mu_0\frac{\partial H_y}{\partial t'}=\frac{\partial}{\partial y'}\left[\gamma\left(E_z-v\mu_0 H_y\right)\right]-\frac{\partial}{\partial z'}\left[\gamma\left(E_y-v\mu_0 H_z\right)\right].(11)$$

Applying the principle of relativity again, for the observer ω, the first equation in $(2')$ will be written:

$$-\mu_0 \frac{\partial H_x'}{\partial t'} = \frac{\partial E_z'}{\partial y'} - \frac{\partial E_u'}{\partial z'}$$

which, compared to (11), leads to the formulas in the theorem statement.

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3. CONSEQUENCES

a) A simple calculation shows that the inverses of formulas (8) and (9) have the same form, except that instead of v appears $-v$. The following relationship is used in the demonstration

$c^{-2} = \mu_0 \varepsilon_0$.

b) if the observer Ω finds only the presence of a magnetic field (possibly even constant), ie $E_x = E_y = E_z = 0$, the observer ω, which moves with velocity v with respect to the first in the direction of the axis $Ox = Ox'$, will also notice the presence of an electric field of non-zero components in the directions Oy and Oz. Indeed, even if $E_x = E_y = E_z = 0$, formulas (8) and (9) give us non-zero values for

$E_x' = 0, E_y' = -\gamma v \mu_0 H_z, E_z' = \gamma v \mu_0 H_x,$

$H_x' = H_x, H_y' = \gamma H_y, H_z' = \gamma H_z.$

c) if the observer Ω finds only the presence of an electric field, ie $H_x = H_y = H_z = 0$, the observer ω will also measure a magnetic field, so an electromagnetic field, of components

$E_x' = E_x, E_y' = \gamma E_y, E_z' = \gamma E_z,$

$H_x' = 0, H_y' = \gamma v \varepsilon_0 E_z, H_z' = -\gamma v \varepsilon_0 E_y.$

All these theoretical results which appear as consequences of formulas (8) and (9) are confirmed by experience.

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