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FOUR-VECTOR QUANTITIES OF RELATIVISTIC ELECTRODYNAMICS

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Abstract. We will present in this paper how the quantities of relativistic electrodynamics can be studied as vectors in relation to the Lorentz transformation group. For this we will introduce two 4-vector quantities, namely 4-charge-current vector and 4-potential and we will also introduce the notion of 4-divergence of a 4-vector field. We will express the continuity equation (charge conservation law) using the components of the 4-load-current vector and D'Alambert's equation for the potential 4-vector of the electromagnetic field. Finally we will demonstrate two properties that are direct consequences of the 4-vector character of the 4- vector charge-current and 4-potential sizes.
Keywords: Minkowski spaces, Lorentz transformations, 4-potential, 4-vector charge-current

1. INTRODUCTION

We will present in this paper how the quantities of relativistic electrodynamics can be studied as vectors in relation to the Lorentz transform group. For this we will introduce two 4-vector quantities, namely the *4-vector charge- current* and the *4-potential*, with the help of which we will be able to concisely write the continuity equation and the Lorentz calibration equation. These 4-vectors can still be used in the construction of other quantities of relativistic electrodynamics, with tensor character.

In order to justify the way of combining the classical, three-dimensional quantities, which leads to the respective 4-dimensional quantities, we will analyze how these quantities behave in relation to the Lorentz transformations.

The Lorentz transform of the load density is similar to the temporal coordinate transform $x^0=ct$. Indeed, the 4-dimensional volume element $\Delta x^0 \Delta x^1 \Delta x^2 \Delta x^3$ is invariant to Lorentz transformations, because based on relativistic effects (see [4]) we have

$$\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}} \wedge \Delta V' = \Delta V\left(1-\frac{v^2}{c^2}\right),$$

where $\Delta V = \Delta x^1 \Delta x^2 \Delta x^3$ is the three-dimensional volume element, therefore $q = \rho \Delta V = constant$ so that the load density ρ convert as a x^0 (respectively t), namely

$$\rho' = \frac{\rho}{\sqrt{1-\frac{v^2}{c^2}}} (1)$$

Lorentz transform of current density $\bar{j} = \rho \bar{v}$ is similar to the Lorentz transform of the three-dimensional position vector that determines where an event occurs. Indeed, writing on components, we have

$$j^k = \rho \frac{dx^k}{dt}, k = 1,2,3. (2)$$ because ρ and t it changes in the same way, it turns out that the rule of changing components j^k is determined by the rule of changing spatial coordinates x^k of events.

2. 4-VECTOR CHARGE- CURRENT

Given these rules for changing charge and current densities with respect to Lorentz transformations and the observation that cρ and j have the same physical dimension is the consideration of the next 4-dimensional physical quantity.

Definition. Is called *4-vector charge-current* physical size $J = (c\rho, \bar{j})$ whose components, in the canonical basis, are

$$J^\mu = \rho \frac{dx^\mu}{dt}, \mu = 0,1,2,3. (3)$$

It is easy to see that $J^0 = c\rho$, because $x^0 = ct$, and $J^k = \rho v^k, k = 1,2,3$, which ensures the vector character of the charge-current quantity in the Minkowski space. The indices were written above to highlight the character of the 4-contrast vector of size $(c\rho, \bar{j})$, determined by the contravariance of the 4-position vector of an event $e = (ct, \bar{r})$.

Property. (4-vector form of the law of conservation of charge). The continuity equation (load conservation law) is expressed, using the components of the 4- vector charge-current, as

$$\sum_{\mu=0}^{3} \frac{\partial J^{\mu}}{\partial x^{\mu}} = 0. \quad (4)$$

Demonstration. In the classical formulation, the continuity equation is written in the form

$$\nabla \bar{j} + \frac{\partial \rho}{\partial t} = 0,$$

so it remains to be seen that

$$\frac{\partial \rho}{\partial t} = \frac{\partial J^{0}}{\partial x^{0}}$$

and

$$\nabla \bar{j} = \frac{\partial j^{1}}{\partial x} + \frac{\partial j^{2}}{\partial y} + \frac{\partial j^{3}}{\partial z} = \sum_{k=1}^{3} \frac{\partial J^{k}}{\partial x^{k}}$$

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Similarity of expression

$$\sum_{\mu=0}^{3} \frac{\partial J^{\mu}}{\partial x^{\mu}}$$

which appears in formula (4) of the continuity equation, with the expression of the divergence $¿\bar{u} = \langle \nabla, \bar{u} \rangle$, as a differential operator on a 3-dimensional vector field $\bar{u}$, justifies the introduction of the following notion.

Definition. We call the 4-divergence of a 4-vector field W, size

$$4 - ¿W = \sum_{\mu=0}^{3} \frac{\partial W^{\mu}}{\partial x^{\mu}}. \quad (5)$$

Linear differential operator

$$\nabla^{¿} = \frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = \sum_{\mu=0}^{3} \frac{\partial}{\partial x^{\mu}}. \quad (6)$$

it is called Hamilton's 4-dimensional operator, or in short, 4-Hamiltonian. Its (covariant) components are

$$\nabla_\mu^{¿}=\frac{\partial}{\partial x^\mu},\mu=0,1,2,3.(7)$$

It is observed that, using these notations, equation (4), of continuity, can be written in the form

$$4-¿(c\rho,\bar{j})=\langle\nabla^{¿},(c\rho,\bar{j})\rangle_{R^4}=\sum_{\mu=0}^{3}\nabla_\mu^{¿}J^\mu=0.$$

D'Alambert's equation in the presence of charges and currents is deduced from Maxwell's equations. Thus, the equations that the fields check $\overline{E}$ and $\overline{H}$ in the presence of density charges ρ and density currents $\bar{j}$ are written in S.I. as

$$¿\,\overline{E}=\frac{1}{\varepsilon_0}\rho$$

$$rot\,\overline{E}=\frac{-\partial\overline{B}}{\partial t}$$

$$¿\,\overline{B}=0(8)$$

$$c^2\,rot\,\overline{B}=\frac{1}{\varepsilon_0}\bar{j}+\frac{\partial\overline{E}}{\partial t}$$

where $\overline{B}=\mu_0\overline{H}$. We will first proceed to solve these equations by introducing the potentials $\overline{A}$ and φ. Thus out of condition $¿\,\overline{B}=0$ we deduce the existence of a vector field $\overline{A}$ so that we have:

$$\overline{B}=rot\,\overline{A}(9)$$

Entering this expression in the second equation of (8) is obtained

$$rot\left(\overline{E}+\frac{\partial\overline{A}}{\partial t}\right)=0,$$

hence the existence of a scalar field φ such that

$$\overline{E}+\frac{\partial\overline{A}}{\partial t}=-grad\varphi.$$

From here we deduce for the field $\overline{E}$ the solution

$$\overline{E}=\frac{-\partial\overline{A}}{\partial t}-grad\varphi(10)$$

ormulas (9) and (10) are considered to be the solutions of Maxwell's equations, without explicit consideration of charges and currents. Let us further determine the equations verified by the potentials $\overline{A}$ and φ considered above, using the other equations (8), thus taking into account

charges and currents. For this we will introduce the expressions for $\overline{B}$ and $\overline{E}$ from formulas (9) and (10) in the fourth equation of (8) and we obtain.

$$c^2 \, rot \, rot \, \overline{A} = \frac{1}{\varepsilon_0} \bar{j} - \frac{\partial^2 \overline{A}}{\partial t^2} - grad \, \frac{\partial \varphi}{\partial t}.$$

Specifying

$rot \, rot \, \overline{A} = grad \div \overline{A} - \Delta \overline{A}$,

the equation is obtained

$$-\Delta \overline{A} + \frac{1}{c^2} \frac{\partial^2 \overline{A}}{\partial t^2} = \mu_0 \bar{j} - grad \left(¿ \, \overline{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} \right),$$

where it was taken into account that $\varepsilon_0 \mu_0 = c^{-2}$. Using Lorentz calibration is considered

$$¿ \, \overline{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} = 0, \quad (11)$$

so using D'Alambert's operator

$$\boxdot = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

we can write

$$\boxdot \, \overline{A} = -\mu_0 \bar{j} \quad (12)$$

To obtain the equation that the scalar potential verifies φ we replace his expression $\overline{E}$ of (10) in the first formula of (8) and we obtain

$$\frac{-\partial}{\partial t} \div \overline{A} - \Delta \varphi = \frac{1}{\varepsilon_0} \rho$$

Replacing here $¿ \, \overline{A}$ according to the Lorentz calibration (11), it results

$$\boxdot \varphi = \frac{-1}{\varepsilon_0} \rho. \quad (13)$$

In conclusion, D'Alambert's equations in the presence of loads and currents are expressed by conditions (12) and (13).

Lorentz transform of potentials $\overline{A}$ and φ is deduced from D'Alambert's equations (12) and (13) taking into account the rules for transforming densities ρ and $\bar{j}$ and operator invariance $\boxdot$. In other words, because the operator $\boxdot$ does not change as a result of a Lorentz

transformation, from equations (12) and (13) we deduce that, compared to such a transformation, φ changes as ρ and $\overline{A}$ changes as $\overline{j}$.

3. 4-POTENTIAL

Regarding the physical dimensions of the potentials $\overline{A}$ and φ, also from the equations (12) and (13) we deduce that $[c\,\varepsilon_0\varphi]=\left[\frac{1}{\mu_0}\overline{A}\right]$, because $[c\rho]=[\overline{j}]$. In conclusion, $[\varphi]=[c\,\overline{A}]$ which justifies the consideration of the next physical quantity.

Definition. We call the 4-potential (quadripotential) of the electromagnetic field 4-vector $A=(\varphi, c\,\overline{A})$ of components

$$A^0=\varphi\wedge A^k=c\,A^k, k=1,2,3 \quad (14)$$

The character of a 4-vector contravariant of size A results from the similar property of 4-vector J, based on the fact that by unifying equations (12) and (13) we can highlight a simple connection between them, namely

$$\boxdot A=-c\,\mu_0 J \quad (15)$$

In this way, the writing of components (14) with higher indices is also justified. Equation (15) represents **D'Alambert's equation for the potential 4-vector** of the electromagnetic field, being in fact the result of assembling in a single 4-dimensional equation, equations (12) and (13)

4-vector form of Lorentz calibration expresses the 4-divergence cancellation of the 4-vector A, that is, the fact that:

$$\nabla^{\wr} A=0. \quad (16)$$

This form of condition (11) results from a direct calculation:

$$\nabla^{\wr} A=\frac{\partial A^0}{\partial x^0}+\frac{\partial A^0}{\partial x^0}+\frac{\partial A^0}{\partial x^0}+\frac{\partial A^0}{\partial x^0}=\frac{1}{c}\frac{\partial \varphi}{\partial t}+c\,div\,\overline{A}=0.$$

The following two properties are direct consequences of the 4-vector character of the quantities J and A.

Proposition. If an inertial observer Ω notes the presence of a charge density at a point in space ρ and current $\overline{j}$ and an electromagnetic field of potentials $\overline{A}$ and φ then another inertial observer ω which moves towards Ω with speed v in the direction of the axis $Ox'=Ox$ will find a

charge density ρ' and current $\bar{j}'$ as well as an electromagnetic field of potentials $\bar{A}'$ and φ' given by the formulas:

$$\begin{cases} j_x^{'}=\gamma\left(j_x-v\rho\right) \\ j_y^{'}=j_y \\ j_z^{'}=j_z \\ \rho^{'}=\gamma\left(\rho-\dfrac{v}{c^2}j_x\right) \end{cases} \quad (17)$$

and respectively

$$\begin{cases} A_x^{'}=\gamma\left(A_x-\dfrac{v}{c^2}\varphi\right) \\ A_y^{'}=A_y \\ A_z^{'}=A_z \\ \varphi^{'}=\gamma\left(\varphi-v\,A_x\right) \end{cases} \quad (18)$$

Demonstration. We take that into account J and A transforms like the 4-vector $\left(x^0=ct, x^1=x, x^2=y, x^3=z\right)$ and we replace $\left(x^0, x^1, x^2, x^3\right)$ with $\left(c\rho, j_x, j_y, j_z\right)$ respectively with $\left(\varphi, c\,A_x, c\,A_y, c\,A_z\right)$ in Lorentz's formulas written in a special form in which it has been replaced t with x^0, namely:

$$\begin{cases} x^{'}=\gamma\left(x-\dfrac{v}{c}x^0\right) \\ y^{'}=y \\ z^{'}=z \\ x^{0\prime}=\gamma\left(x^0-\dfrac{v}{c}x\right) \end{cases}$$

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Proposition. The following sizes

$c^2\rho^2-\langle\bar{j},\bar{j}\rangle$

and

$\varphi^2-c^2\langle\bar{A},\bar{A}\rangle$

are invariants of electromagnetic fields to changes in inertial reference systems.

Demonstration. In the inner product $\langle .,. \rangle$ of the Minkowski space R^4 we have

$$(J,J)=c^2\rho^2-(\bar{j},\bar{j}) \quad (19)$$

and

$$(A,A)=\varphi^2-c^2(\overline{A},\overline{A}). \quad (20)$$

So it remains to keep in mind that the size of any 4-vector is an invariant with respect to Lorentz transformations.

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