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DETERMINATION OF SOME FUNDAMENTAL INVARIANTS OF THE ELECTROMAGNETIC FIELD

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Abstract*. In this paper we will present a theorem that allows obtaining the fundamental invariants of the electromagnetic field starting from the expression of the 4-tensor of the electromagnetic field with the help of quantities that represent the solutions of Maxwell's equations. Finally, we will present some properties of the electromagnetic field which are simple consequences of the theorem,*

Keywords: Minkowski spaces, Maxwell's equations, electromagnetic field

1. INTRODUCTION

We will consider the 4-tensor of the electromagnetic field. This is a physical quantity whose covariant components are

$$F_{\mu v} = \nabla_{\mu}^{*}\mathcal{A}_{v} - \nabla_{v}^{*}\mathcal{A}_{\mu} = \frac{\partial \mathcal{A}_{v}}{\partial x^{\mu}} - \frac{\partial \mathcal{A}_{\mu}}{\partial x^{v}} \qquad (1)$$

The utility of this tensor lies in the fact that it gives us the opportunity to group the components of the fields $\bar{E}$ and $\bar{B}$, which represent the solutions of Maxwell's equations (see [3]), so that we can deduce how they change following a Lorentz transformation.

For the fields $\bar{E}$ and $\bar{B}$ we have the following formulas (see [5])

$$\begin{cases} E_x = \dfrac{\partial \mathcal{A}_1}{\partial x^0} - \dfrac{\partial \mathcal{A}_0}{\partial x^1} \\ E_y = \dfrac{\partial \mathcal{A}_2}{\partial x^0} - \dfrac{\partial \mathcal{A}_0}{\partial x^2} \\ E_z = \dfrac{\partial \mathcal{A}_3}{\partial x^0} - \dfrac{\partial \mathcal{A}_0}{\partial x^3} \end{cases} \qquad (2)$$

and respectively:

$$\begin{cases} cB_x = \dfrac{\partial \mathcal{A}_2}{\partial x^3} - \dfrac{\partial \mathcal{A}_3}{\partial x^2} \\ cB_y = \dfrac{\partial \mathcal{A}_3}{\partial x^1} - \dfrac{\partial \mathcal{A}_1}{\partial x^3} \\ cB_z = \dfrac{\partial \mathcal{A}_1}{\partial x^2} - \dfrac{\partial \mathcal{A}_2}{\partial x^1} \end{cases} \qquad (3)$$

Expressing the 4-tensor of the electromagnetic field using quantities $\bar{E}$ and $\bar{B}$ is obtained by identifying the components $F_{\mu v}$ from the formulas (1) in the formulas (2) and (3).

These values are written in an array of the form

$$(F_{\mu v}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix} \qquad (4)$$

From this expression also follows the fact that the elements are uniquely determined by the fields $\bar{E}$ and $\bar{B}$, (in each reference system), although the potentials φ and $\bar{A}$, so also the 4-potential $\mathcal{A}$, are not uniquely determined.

The Lorentz transformation of the fields $\bar{E}$ and $\bar{B}$ can be easily obtained from the transformation law of the components of the 4-tensor $(F_{\mu v})$, given by the expression(4) namely

$$F'_{\mu v} = L^{\alpha}_{\mu} L^{\beta}_{v} F_{\alpha\beta} \, . \qquad (5).$$

where L^{α}_{μ} represents the components of the Lorentz transformation matrix of the algebraic basis. In the case of special Lorentz transformations we have:

$$L^{\alpha}_{\mu} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (6)$$

where we noted $\beta = vc^{-1}$ and $\gamma = (1-\beta^2)^{-1/2}$.

2. FUNDAMENTAL INVARIANTS OF THE ELECTROMAGNETIC FIELD

Knowing how to transform the fields allows the fundamental invariants of the electromagnetic field to be obtained, as seen in the following theorem.

Theorem. For any inertial observer, the quantities

$$c^2\langle \bar{B}, \bar{B}\rangle - \langle \bar{E}, \bar{E}\rangle = c^2B^2 - E^2 \qquad (7)$$

$$\langle \bar{E}, \bar{B}\rangle = \bar{E}\cdot\bar{B} \qquad (8)$$

have the same value (so they are Lorentz invariant).

Demonstration. The tensor calculus, however, offers us the means to obtain invariants, which essentially boil down to the formation of combinations between components so that we no longer have free indices, i.e. the respective expression is a constant. In particular, to tensorial prove formula (7), let us note that, using the contravariant components $F^{\mu v}$, we obtain $F_{\mu v}F^{\mu v}$-invariant. It remains to take into account the rule of passing to contravariant components (see [5]), according to which we obtain

$$(F^{\mu v}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix} \qquad (9)$$

and explain the amount:

$$F_{\mu v}F^{\mu v} = -2E_x^2 - 2E_y^2 - 2E_z^2 + 2c^2B_z^2 + 2c^2B_y^2 + 2c^2B_x^2 = 2[c^2\langle\ \ \rangle]$$

To prove the invariance of expression (8) we will appeal to the unitary Levi-Civita tensor, completely antisymmetric, of contravariant components:

$$\varepsilon^{\mu v\sigma\omega} = sign\begin{pmatrix} 0 & 1 & 2 & 3 \\ \mu & v & \sigma & \omega \end{pmatrix}.$$

The components of this tensor are: +1 for even permutation, -1 for odd permutation, and 0 for any pair of even indices. The 4-tensor character of this quantity results from the fact that for any Lorentz transformation we have:

$$\det L = 1 = \varepsilon^{\mu v\sigma\omega}L_\mu^0 L_v^1 L_\sigma^2 L_\omega^3 .$$

and $\varepsilon^{0123} = 1$. Using this tensor we obtain the invariant:

$$\varepsilon^{\mu v\sigma\omega}F_{\mu v}F_{\sigma\omega} = -8c\left[E_xB_x + E_yB_y + E_zB_z\right] = -8c\bar{E}\cdot\bar{B} .$$

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Properties of the electromagnetic field. From the invariance of expressions (7) and (8) therefore as simple consequences of the previous theorem, the following properties of the electromagnetic field result:

1. If for an inertial observer the (three-dimensional) vectors $\bar{E}$ and $\bar{B}$ are orthogonal, then they are orthogonal for any other inertial observer.

2. If for an inertial observer the vectors $c\bar{B}$ and $\bar{E}$ have the same magnitude (Euclidean), then any other inertial observer finds them to have the same magnitude; is the case when $c^2B^2 - E^2 = 0$.

3. The fact that the angle (in the Euclidean sense) between $\bar{E}$ and $\bar{B}$ is acute (respectively obtuse) represents a Lorentz invariant; the sign of the product $\bar{E} \cdot \bar{B}$ is maintained.

4. The inequality between the magnitudes of the vectors $c\bar{B}$ and $\bar{E}$ is a Lorentz invariant; the sign of the expression $c^2B^2 - E^2$ is the same for all inertial observers.

5. In the hypothesis that $c^2B^2 - E^2 \neq 0$ if for an inertial observer Ω the fields $\bar{E}$ and $\bar{B}$ are orthogonal, there is another inertial observer ω for which $\bar{E}' = 0$ (or $\bar{B}' = 0$ depending on the sign of the expression $c^2B^2 - E^2$). Reciprocally, if for an inertial observer ω we have $\bar{E} = 0$ (or $\bar{B} = 0$), then for any other inertial observer Ω these fields are orthogonal.

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